Characteristic polynomial, eigenvalues, and eigenvectors

For each of the following matrices, perform the following tasks:

1. Find the characteristic polynomial
2. Find the eigenvalues and the associated eigenvectors

a)

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

b)

$$\begin{pmatrix} 2 & -1 & -1 \\ 4 & -8 & -6 \\ -4 & 11 & 9 \end{pmatrix}$$

Solution

a) Characteristic polynomial

The characteristic polynomial of A is obtained by computing

$$p(\lambda) = \det(A - \lambda I)$$

In this case,

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}$$

Therefore,

$$p(\lambda) = \det \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - (3 \cdot 2)$$

Expanding the terms:

$$p(\lambda) = (1-\lambda)(2-\lambda) - 6 = (2 - \lambda - 2\lambda + \lambda^2) - 6 = \lambda^2 - 3\lambda - 4$$

We can factorize it as:

$$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

Thus, the characteristic polynomial of A is

$$p(\lambda) = \lambda^2 - 3\lambda - 4$$

Eigenvalues and eigenvectors

From the factorization above, we obtain the eigenvalues:

$$\lambda_1 = 4, \quad \lambda_2 = -1$$

Eigenvalue $\lambda_1 = 4$

To find the associated eigenvector, we solve

$$(A - 4I)\mathbf{v} = 0$$

In matrix form:

$$A - 4I = \begin{pmatrix} 1-4 & 2 \\ 3 & 2-4 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$$

Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. The resulting linear system is:

$$\begin{cases} -3x + 2y = 0 \\ 3x - 2y = 0 \end{cases}$$

Both equations are equivalent. From the first one,

$$-3x + 2y = 0 \quad \Longrightarrow \quad 2y = 3x \quad \Longrightarrow \quad y = \frac{3}{2}x$$

Choosing, for example, $x = 2$, we get $y = 3$. A corresponding eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Eigenvalue $\lambda_2 = -1$

For the second eigenvalue, we solve

$$(A - (-1)I)\mathbf{v} = 0 \quad \Longrightarrow \quad (A + I)\mathbf{v} = 0$$

In matrix form:

$$A + I = \begin{pmatrix} 1+1 & 2 \\ 3 & 2+1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. The resulting linear system is:

$$\begin{cases} 2x + 2y = 0 \\ 3x + 3y = 0 \end{cases}$$

From the first equation,

$$2x + 2y = 0 \quad \Longrightarrow \quad x + y = 0 \quad \Longrightarrow \quad y = -x$$

Choosing, for example, $x = 1$, we get $y = -1$. A corresponding eigenvector is:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b) Characteristic polynomial

The characteristic polynomial of A is obtained by computing

$$p(\lambda) = \det(A - \lambda I)$$

In this case, considering

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 4 & -8 & -6 \\ -4 & 11 & 9 \end{pmatrix},$$

we have

$$A - \lambda I = \begin{pmatrix} 2-\lambda & -1 & -1 \\ 4 & -8-\lambda & -6 \\ -4 & 11 & 9-\lambda \end{pmatrix}$$

Therefore,

$$p(\lambda) = \det \begin{pmatrix} 2-\lambda & -1 & -1 \\ 4 & -8-\lambda & -6 \\ -4 & 11 & 9-\lambda \end{pmatrix}$$

Expanding along the first row, we obtain:

$$p(\lambda) = (2-\lambda)(\lambda^2 - \lambda - 6)$$

Noting that

$$\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$$

Thus, the characteristic polynomial of A is

$$p(\lambda) = (2-\lambda)(\lambda - 3)(\lambda + 2)$$

Eigenvalues and eigenvectors

From the factorization above, we obtain the eigenvalues:

$$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = -2$$

Eigenvalue $\lambda_1 = 2$

To find the associated eigenvector, we solve

$$(A - 2I)\mathbf{v} = 0$$

In matrix form,

$$A - 2I = \begin{pmatrix} 2-2 & -1 & -1 \\ 4 & -8-2 & -6 \\ -4 & 11 & 9-2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 4 & -10 & -6 \\ -4 & 11 & 7 \end{pmatrix}$$

Let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The resulting system is:

$$\begin{cases} -y - z = 0 \\ 4x - 10y - 6z = 0 \\ -4x + 11y + 7z = 0 \end{cases}$$

From the first equation, we have $y = -z$. Substituting into the second:

$$4x - 10(-z) - 6z = 4x + 10z - 6z = 4x + 4z = 0 \quad \Longrightarrow \quad x = -z$$

Choosing, for example, $z = 1$, we obtain:

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Eigenvalue $\lambda_2 = 3$

To find the associated eigenvector, we solve

$$(A - 3I)\mathbf{v} = 0$$

In matrix form,

$$A - 3I = \begin{pmatrix} 2-3 & -1 & -1 \\ 4 & -8-3 & -6 \\ -4 & 11 & 9-3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ 4 & -11 & -6 \\ -4 & 11 & 6 \end{pmatrix}$$

Let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The resulting system is:

$$\begin{cases} -x - y - z = 0 \\ 4x - 11y - 6z = 0 \\ -4x + 11y + 6z = 0 \end{cases}$$

From the first equation, $x + y + z = 0$, which means $x = -y - z$. Substituting into the second:

$$4(-y - z) - 11y - 6z = -4y - 4z - 11y - 6z = -15y - 10z = 0$$

which simplifies to:

$$15y + 10z = 0 \quad \Longrightarrow \quad y = -\frac{2}{3}z$$

Then,

$$x = -\left(-\frac{2}{3}z\right) - z = \frac{2}{3}z - z = -\frac{1}{3}z$$

Choosing $z = 3$, we obtain:

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$$

Eigenvalue $\lambda_3 = -2$

To find the associated eigenvector, we solve

$$(A + 2I)\mathbf{v} = 0$$

In matrix form,

$$A + 2I = \begin{pmatrix} 2+2 & -1 & -1 \\ 4 & -8+2 & -6 \\ -4 & 11 & 9+2 \end{pmatrix} = \begin{pmatrix} 4 & -1 & -1 \\ 4 & -6 & -6 \\ -4 & 11 & 11 \end{pmatrix}$$

Let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The resulting system is:

$$\begin{cases} 4x - y - z = 0 \\ 4x - 6y - 6z = 0 \\ -4x + 11y + 11z = 0 \end{cases}$$

From the first equation, we obtain $4x = y + z$, which means $x = \frac{y+z}{4}$. Substituting into the second:

$$4\left(\frac{y+z}{4}\right) - 6y - 6z = y + z - 6y - 6z = -5y - 5z = 0$$

which implies that $y + z = 0$ or $y = -z$. Then, $x = 0$. A corresponding eigenvector is:

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$